

Painlevé V and the distribution function of a discontinuous linear statistics in the Laguerre Unitary Ensembles

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Abstract

In this paper we study the characteristic or generating function of a certain discontinuous linear statistics of the Laguerre unitary ensembles and show that this is a particular fifth Painlevé transcendant in the variable t , the position of the discontinuity.

The proof of the ladder operators adapted to orthogonal polynomial with discontinuous weight announced sometime ago [13] is presented here, followed by the non-linear difference equations satisfied by two auxiliary quantities and the derivation of the Painlevé equation.

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1 Introduction

In the theory of random matrix ensembles with unitary symmetry the real eigenvalues $\{x_j\}_{j=1}^n$ have the joint probability distribution

$$P(x_1, \dots, x_n) dx_1 \dots dx_n = \frac{1}{n! D_n} \prod_{1 \leq j < k \leq n} (x_k - x_j)^2 \prod_{l=1}^n w_0(x_l) dx_l, \quad (1.1)$$

where $w_0(x)$ with $x \in [a, b]$ say, is strictly positive and satisfies a Lipschitz condition and has finite moments, that is, the existence of the integrals,

$$\int_a^b x^j w_0(x) dx, \quad j \in \{0, 1, 2, \dots\}.$$

Here D_n is the normalization constant

$$D_n[w_0] = \frac{1}{n!} \int_{[a,b]^n} \prod_{1 \leq j < k \leq n} (x_k - x_j)^2 \prod_{l=1}^n w_0(x_l) dx_l, \quad (1.2)$$

so that

$$\int_{[a,b]^n} P(x_1, \dots, x_n) dx_1 \dots dx_n = 1. \quad (1.3)$$

We include the cases where a may be $-\infty$ and/or b may be ∞ . For a comprehensive study of the theory of random matrices see [20].

A linear statistics is a linear sum of a certain function g of the random variable x_j :

$$\sum_{k=1}^n g(x_k). \quad (1.4)$$

The generating function of such a linear statistics is, by definition the average of $\exp(\lambda \sum_k g(x_k))$, with respect to the joint probability distribution (1.1), where λ is a parameter, reads,

$$\int_{[a,b]^n} \exp \left[\lambda \sum_{k=1}^n g(x_k) \right] P(x_1, \dots, x_n) \prod_{k=1}^n dx_k. \quad (1.5)$$

More generally, we can consider

$$\int_{[a,b]^n} \left[\prod_{k=1}^n f(x_k) \right] P(x_1, \dots, x_n) \prod_{k=1}^n dx_k \quad (1.6)$$

If f is a "smooth" function then asymptotic formulas for large n for the characteristic functions have been obtained for the Hermite case, where $w_0(x) = e^{-x^2}$, $x \in (-\infty, \infty)$

by Kac [16] and Akhiezer[1] and generalized by many authors. See [8] for a history of this problem. These results are continuous analogs of the classical Szegő limit theorem on Toeplitz determinants

In the Laguerre case where $w_0(x) = x^\alpha e^{-x}$, $x \in [0, \infty)$ an analogous formula was found recently for smooth functions in [4]. However, results for the situations where f has discontinuities are harder to come by. We mention here the original studies in [6] where f has several discontinuities and which corresponds to the Hermite case. More general results can be found in [9, 10, 7] and [21]. Also, results that correspond to $\alpha = \pm 1/2$ and large n appear in [5].

In this paper we investigate the case where n is finite, and f is constant except for a jump at $t \in [0, \infty)$, and is of the form

$$f(x, t) = A + B\theta(x - t) \quad (1.7)$$

where $\theta(x)$ is one for $x > 0$ and zero otherwise and $A \geq 0$ and $B > 0$. In the special case of linear statistics the function g will take the form

$$g(x, t) := \theta(x - t) \ln \left(1 + \frac{\beta}{2} \right) + \theta(t - x) \ln \left(1 - \frac{\beta}{2} \right), \quad (1.8)$$

where $-1 < \frac{\beta}{2} < 1$. This corresponds to a function f where

$$f(x, t) = \left(1 - \frac{\beta}{2} \right)^\lambda + \left[\left(1 + \frac{\beta}{2} \right)^\lambda - \left(1 - \frac{\beta}{2} \right)^\lambda \right] \theta(x - t)$$

that is

$$A = \left(1 - \frac{\beta}{2} \right)^\lambda$$

and

$$B = \left(1 + \frac{\beta}{2} \right)^\lambda - \left(1 - \frac{\beta}{2} \right)^\lambda.$$

We also point out that if $A = 0$ and $B = 1$ then we have the important case where we are computing the probability that all the eigenvalues are in the interval $[t, \infty)$. This case of course is what leads to the now well-known Tracy-Widom laws. More will be said about this later.

Our main tool will be to use the theory of orthogonal polynomials. Previously, in random matrix theory one made use of the orthogonal polynomials associated to the weight that defined the ensemble. Fundamental quantities were then described in terms of Fredholm determinants. While both the authors are very fond of determinants, in this work, we do not consider Fredholm determinants. Instead we consider the polynomials that are orthogonal to the perturbed weight, that is a regular or "nice" weight multiplied by the discontinuous factor given in (1.7). In this manner we are able to use the results of the orthogonal polynomials to derive equations associated with the various statistics of interest.

The idea is that we write the multiple integral in (1.6) as a Hankel determinant. We then need to know information about the norms of the orthogonal polynomials. To understand this we need to know something about the recursion coefficients of the polynomials. This will lead us naturally to another pair of auxiliary quantities that depend on t and n . In the paper they are called r_n and R_n . Using these auxiliary quantities we are able to produce the second order non-linear differential equations satisfied by $S_n = 1 - 1/R_n$ which turns out to be a particular fifth Painlevé transcendent, in addition to the Jimo-Miwa-Okamoto σ form [15] satisfied by the logarithmic derivative of the Hankel determinant with respect to t . We also wish to emphasize that the logarithmic derivative of the Hankel determinant can be computed very naturally in terms of our quantity $r_n(t)$ and its derivative and the relationships between these quantities arise naturally using this approach.

We also derive a discrete version of the σ form of a nonlinear second order difference equation satisfied by the same logarithmic derivative. Our computations show that in fact the values of our generalized polynomials at the end points of the intervals are intimately related to the resolvent kernels found in the standard approach of Tracy and Widom. This is really not surprising, since we are all starting with the same multiple integral. Rather, our point is that computations can all be made by using only the very basic theory of orthogonal polynomials.

The Painlevé equation can be found in [27]. The second order difference equation [(4.29), **Theorem 8**], as far as we know is a new equation.

In the next section the proof for a pair of ladder operators, and the associated supplementary conditions adapted to orthogonal polynomials with discontinuous weights which was announced sometime ago [13] will be provided. In section 3, a system of difference equations satisfied by two auxiliary quantities r_n and R_n (these will ultimately determine the recurrence coefficients for the orthogonal polynomials) are derived. In section 4 we derive a second order non-linear differential equation which turns out to be a particular fifth Painlevé transcendent. In the process we identify that the quantity

$$S_n(t) := 1 - \frac{1}{R_n(t)},$$

to be such an equation. Furthermore we show that the logarithmic derivative of the generating function

$$H_n(t) := t \frac{d}{dt} \ln G(n, t) = t \frac{d}{dt} \ln D_n(t)$$

satisfies both the continuous and discrete σ form of Painlevé V.

2 Ladder operators and supplementary conditions

According the general theory of orthogonal polynomials of one variable, for a generic weight w , the normalization constant (1.2) has the two more alternative representations

$$D_n[w] := \det(\mu_{i+j})_{i,j=0}^{n-1} := \det \left(\int_a^b x^{i+j} w(x) dx \right)_{i,j=0}^{n-1} \quad (2.1)$$

$$= \prod_{j=0}^{n-1} h_j, \quad (2.2)$$

where the determinant of the moment matrix (μ_{i+j}) is the Hankel determinant. Here $\{h_j\}_{j=0}^n$ is the square of the L^2 norm of the sequence of (monic-)polynomials $\{P_j(x)\}_{j=0}^n$ orthogonal with respect to w over $[a, b]$;

$$\int_a^b P_i(x) P_j(x) w(x) dx = \delta_{i,j} h_j. \quad (2.3)$$

Therefore with reference to (1.2) and (1.5) the quantity that we need to compute is

$$\mathbf{G}(t, n) = \frac{D_n[w]}{D_n[w_0]} = \frac{\prod_{i=0}^{n-1} h_i(t)}{\prod_{i=0}^{n-1} h_i},$$

where $w(x, t) := x^\alpha e^{-x}(A + B\theta(x - t))$ and $h_k(t)$ is defined by

$$\int_0^\infty \{P_k(x)\}^2 (A + B\theta(x - t)) x^\alpha e^{-x} dx = h_k(t). \quad (2.4)$$

We also denote

$$D_n(t) := D_n[w(\cdot, t)].$$

This leads to the generic problem of the characterization of polynomials orthogonal with respect to "smooth" weights $w_0(x)$ perturbed by a jump factor where the discontinuity is at t . So if we write

$$w_J(x, t) := A + B\theta(x - t), \quad A \geq 0, \quad A + B > 0 \quad (2.5)$$

then

$$\int_a^b P_i(x) P_j(x) w_0(x) w_J(x, t) dx = \delta_{i,j} h_j(t). \quad (2.6)$$

It follows from the orthogonality relations that,

$$zP_n(z) = P_{n+1}(z) + \alpha_n(t)P_n(z) + \beta_n(t)P_{n-1}(z). \quad (2.7)$$

This three term recurrence relations, together with the "initial" conditions,

$$P_0(z) = 1, \quad \beta_0 P_{-1}(z) = 0,$$

generates the monic polynomials,

$$P_n(z) = z^n + \mathbf{p}_1(n, t)z^{n-1} + \dots \quad (2.8)$$

the first two of which are

$$\begin{aligned} P_0(z) &= 1 \\ P_1(z) &= z - \alpha_0(t) = z - \frac{\mu_1(t)}{\mu_0(t)}. \end{aligned} \quad (2.9)$$

Note that due to the t dependence of the weight, the coefficients of the polynomials and the recurrence coefficients α_n and β_n also depend on t the position of the jump. However, unless it is required we do not display the t dependence.

From (2.7) and (2.8), we find, for $n \in \{0, 1, 2, \dots\}$

$$\begin{aligned} \alpha_n &= \mathbf{p}_1(n, t) - \mathbf{p}_1(n+1, t), \\ \sum_{j=0}^{n-1} \alpha_j &= -\mathbf{p}_1(n, t) \end{aligned} \quad (2.10)$$

where $\mathbf{p}_1(0, t) := 0$.

From (2.6) and (2.7) we have the well-known strictly positive expression,

$$\beta_n := \frac{h_n}{h_{n-1}}. \quad (2.11)$$

Another consequence of the recurrence relation is the Christoffel-Darboux formula

$$\sum_{k=0}^{n-1} \frac{P_k(x)P_k(y)}{h_k} = \frac{P_n(x)P_{n-1}(y) - P_n(y)P_{n-1}(x)}{h_{n-1}(x-y)}. \quad (C-D)$$

The above basic information about orthogonal polynomials can be found in [26].

In this section, we give an account of a recursive algorithm for the determination of the α_n , β_n for a given weight. This is based on a pair of ladder operators and the associated supplementary conditions to be denoted as (S_1) and (S_2) . For an general "smooth" weight the lowering and raising operators has been derived by many authors [3, 11, 12, 24]. We should like to note here A.P. Magnus's contribution to this formalism [17, 18, 19]. Indeed, we have been motivated by the investigation of [19] where he obtained the large n behavior of the recurrence coefficients of a generalization of the Jacobi polynomials in which the standard Jacobi weight is perturbed by a "line" analogue to the Fisher-Hartwig singularity. We end the discussion about the ladder operators with the remark that the supplementary conditions for orthogonal polynomials on the unit circle was found in [2] and have been used to compute explicitly the Toeplitz determinants with Fisher-Hartwig symbols.

The lemma below gives a detailed proof of the ladder operators in the discontinuous case where the results were announced sometime ago [13].

Lemma 1 Let $w_0(x)$, $x \in [a, b]$ be a smooth weight function where the associated moments,

$$\int_a^b x^j w_0(x) dx, \quad j \in \{0, 1, 2, \dots\} \quad (2.12)$$

of all order exist.

Let $w_0(a) = w_0(b) = 0$, and $v_0(x) := -\ln w_0(x)$.

The lowering and raising operators for polynomials orthogonal with respect to

$$w(x) := w_0(x)w_J(x, t),$$

are

$$P'_n(z) = -B_n(z)P_n(z) + \beta_n A_n(z)P_{n-1}(z), \quad (2.13)$$

$$P'_{n-1}(z) = [B_n(z) + v'_0(z)]P_{n-1}(z) - A_{n-1}(z)P_n(z), \quad (2.14)$$

where

$$A_n(z) := \frac{R_n(t)}{z-t} + \frac{1}{h_n} \int_a^b \frac{v'_0(z) - v'_0(y)}{z-y} P_n^2(y) w(y) dy \quad (2.15)$$

$$B_n(z) := \frac{r_n(t)}{z-t} + \frac{1}{h_{n-1}} \int_a^b \frac{v'_0(z) - v'_0(y)}{z-y} P_n(y) P_{n-1}(y) w(y) dy \quad (2.16)$$

$$R_n(t) := B \frac{w_0(t)}{h_n(t)} \{P_n(t, t)\}^2 \quad (2.17)$$

$$r_n(t) := B \frac{w_0(t)}{h_{n-1}(t)} P_n(t, t) P_{n-1}(t, t). \quad (2.18)$$

where

$$P_n(t, t) := P_n(z, t) \Big|_{z=t}.$$

Here $\ln w_0(x)$, is well defined since $w_0(x)$ is suppose to be strictly positive for $x \in [a, b]$.

Proof: We start from

$$P'_n(z) = \sum_{k=0}^{n-1} C_{nk} P_k(z),$$

where C_{nk} is determined from the orthogonality relations,

$$C_{nk} = \frac{1}{h_k} \int_a^b P'_n(y) P_k(y) w(y) dy.$$

Therefore

$$\begin{aligned}
P'_n(z) &= \sum_{k=0}^{n-1} \frac{P_k(z)}{h_k} \int_a^b P'_n(y) P_k(y) w(y) dy \\
&= - \sum_{k=0}^{n-1} \int_a^b \frac{P_k(z)}{h_k} P_n(y) \{P'_k(y) w(y) + P_k(y) [B\delta(y-t)w_0(y) + w'_0(y)w_J(y,t)]\} dy \\
&= - \int_a^b P_n(y) \sum_{k=0}^{n-1} \frac{P_k(z)P_k(y)}{h_k} \left[B w_0(y)\delta(y-t) + \frac{w'_0(y)}{w_0(y)} w(y) \right] dy \\
&= - \int_a^b P_n(y) \sum_{k=0}^{n-1} \frac{P_k(z)P_k(y)}{h_k} \{B w_0(y)\delta(y-t) + [\mathbf{v}'_0(z) - \mathbf{v}'_0(y)]w(y)\} dy \\
&= - \int_a^b P_n(y) \frac{P_n(z)P_{n-1}(y) - P_n(y)P_{n-1}(z)}{h_{n-1}(z-y)} \{B\delta(y-t)w_0(y) + [\mathbf{v}'_0(z) - \mathbf{v}'_0(y)]w(y)\} dy
\end{aligned}$$

where we have used integration by parts, (C-D), the definition of \mathbf{v}_0 , (2.11) and that

$$\int_a^b P_n(y) P_k(y) w(y) dy = 0, \quad k = 0, 1, 2, \dots, n-1,$$

to arrive at the above. A little simplification produces (2.15) and (2.16) follows from straight forward application of the recurrence relations. \square

Remark 1. If $w_0(a) \neq 0$, $w_0(b) \neq 0$, the terms

$$w(y) \frac{\{P_n(y,t)\}^2}{h_n(t)(z-y)} \Big|_{y=a}^b \quad \text{and} \quad w(y) \frac{P_n(y,t)P_{n-1}(y,t)}{h_{n-1}(t)(z-y)} \Big|_{y=a}^b$$

are to be added into the definition of $A_n(z)$ and $B_n(z)$ respectively.

Remark 2. If there are several jumps at t_1, \dots, t_N then the first term of (2.15) and (2.16) should be replaced by

$$\begin{aligned}
&\sum_{j=1}^N \frac{R_{n,j}(t_j; t)}{z - t_j} \\
&\sum_{j=1}^N \frac{r_{n,j}(t_j; t)}{z - t_j}
\end{aligned}$$

where

$$\begin{aligned}
R_{n,j}(t_j; t) &:= B_j \frac{w_0(t_j)}{h_n(t)} \{P_n(t_j; t)\}^2 \\
r_{n,j}(t_j; t) &:= B_j \frac{w_0(t_j)}{h_{n-1}(t)} P_n(t_j; t) P_{n-1}(t_j; t) \\
t &:= (t_1, \dots, t_N).
\end{aligned}$$

As in the case of the smooth weight the "coefficients" $A_n(z)$ and $B_n(z)$ that appear in the ladder operators satisfy two identities valid for all $z \in \mathbb{C} \cup \{\infty\}$, which we gather in the next lemma.

Lemma 2 *The functions $A_n(z)$ and $B_n(z)$ satisfy the following identities which hold for all z :*

$$B_{n+1}(z) + B_n(z) = (z - \alpha_n)A_n(z) - \mathbf{v}'_0(z) \quad (S_1)$$

$$1 + (z - \alpha_n)[B_{n+1}(z) - B_n(z)] = \beta_{n+1}A_{n+1}(z) - \beta_n A_{n-1}(z) \quad (S_2)$$

Proof: By a direct computation using the definition of $A_n(z)$ and $B_n(z)$. \square

It turns out that a suitable combination of (S_1) and (S_2) produces an identity involving $\sum_{j=0}^{n-1} A_j(z)$, from which further insight into the recurrence coefficients may be gained.

Lemma 3 *$A_n(z)$, $B_n(z)$ and $\sum_{j=0}^{n-1} A_j(z)$ satisfy the identity*

$$[B_n(z)]^2 + \mathbf{v}'_0(z) B_n(z) + \sum_{j=0}^{n-1} A_j(z) = \beta_n A_n(z) A_{n-1}(z) \quad (S'_2)$$

Proof: Multiply (S_2) by $A_n(z)$ and replace $(z - \alpha_n)A_n(z)$ in the resulting equation by $B_{n+1}(z) + B_n(z) + \mathbf{v}'_0(z)$. See (S_1) . This leads to

$$[B_{n+1}(z)]^2 - [B_n(z)]^2 + \mathbf{v}'_0(z)[B_{n+1}(z) - B_n(z)] + A_n(z) = \beta_{n+1}A_{n+1}(z)A_n(z) - \beta_n A_n(z)A_{n-1}(z).$$

Taking a telescopic sum of the above equation from 0 to $n-1$ with the "initial" conditions, $B_0(z) = 0$ and $\beta_0 A_{-1}(z) = 0$, we have (S'_2) . \square

Let $y = P_n(z)$ we find by eliminating $P_{n-1}(z)$ from the raising and lowering operators, the second order differential equation

Lemma 4

$$y''(z) - \left(\mathbf{v}'_0(z) + \frac{A'_n(z)}{A_n(z)} \right) y'(z) + \left(B'_n(z) - B_n(z) \frac{A'_n(z)}{A_n(z)} + \sum_{j=0}^{n-1} A_j(z) \right) y(z) = 0. \quad (2.19)$$

Proof: By a straight forward computation using (2.15), (2.16) and (S'_2) . \square

Recalling (2.17) and (2.18) we note that if $\mathbf{v}'_0(z)$ is rational in z then the difference kernel, $[\mathbf{v}'_0(z) - \mathbf{v}'_0(y)]/(z - y)$ is rational in z and y . Consequently (S_1) and (S'_2) may be put to good use to obtain a system of difference equations satisfied by the auxiliary quantities R_n and r_n and the recurrence coefficients α_n and β_n . This will be clear in the next section.

3 Recurrence coefficients and difference equations.

For the problem at hand,

$$w_0(x) = x^\alpha e^{-x}, \quad x \in [0, \infty),$$

$$v_0(x) := -\ln w_0(x) = -\alpha \ln x + x$$

and for $\alpha > 0$, $w_0(0) = 0$. Note that $w(\infty) = 0$. An easy computation gives,

$$\frac{v'_0(z) - v'_0(y)}{z - y} = \frac{\alpha}{zy}.$$

Using these and integration by parts we have the following

Lemma 5

$$A_n(z) = \frac{R_n(t)}{z - t} + \frac{1 - R_n(t)}{z}, \quad (3.1)$$

$$B_n(z) = \frac{r_n(t)}{z - t} - \frac{n + r_n(t)}{z}, \quad (3.2)$$

where

$$R_n(t) := B w_0(t) \frac{\{P_n(t, t)\}^2}{h_n(t)} \quad (3.3)$$

$$r_n(t) := B w_0(t) \frac{P_n(t, t) P_{n-1}(t, t)}{h_{n-1}(t)}. \quad (3.4)$$

Proof: Through integration by parts we find,

$$\alpha \int_0^\infty y^{\alpha-1} e^{-y} w_J(y; t) \{P_n(y, t)\}^2 dy = h_n(t) - B w_0(t) \{P_n(t, t)\}^2 \quad (3.5)$$

$$\begin{aligned} \alpha \int_0^\infty y^{\alpha-1} e^{-y} w_J(y; t) P_n(y, t) P_{n-1}(y, t) dy &= -n h_{n-1}(t) \\ &- B w_0(t) P_n(t, t) P_{n-1}(t, t), \end{aligned} \quad (3.6)$$

and we have used the fact that

$$\frac{\partial}{\partial x} P_n(x, t) = n P_{n-1}(x, t) + \text{lower degree}$$

to arrived at (3.6). From (3.5) and (3.6) and the definitions of $A_n(z)$ and $B_n(z)$, (3.1)—(3.4) follows. \square

Substituting (3.1) and (3.2) into (S_1) we find by equating the residues

$$r_{n+1} + r_n = R_n(t - \alpha_n) \quad (3.7)$$

$$-(r_{n+1} + r_n) = 2n + 1 + \alpha - \alpha_n(1 - R_n). \quad (3.8)$$

Lemma 6

$$\alpha_n = 2n + 1 + \alpha + tR_n \quad (3.9)$$

$$r_{n+1} + r_n = R_n(t - \alpha_n) \quad (3.10)$$

Proof: (3.7)+(3.8) implies (3.9) and we restate (3.7) as (3.10). \square

Substituting (3.1) and (3.2) into (S'_2) , we find, after some elementary but messy computations,

$$\begin{aligned} [B_n(z)]^2 + v'_0(z) B_n(z) &+ \sum_{j=0}^{n-1} A_j(z) = \frac{r_n^2}{(z-t)^2} + \frac{(n+r_n)(\alpha+n+r_n)}{z^2} \\ &+ \frac{\sum_{j=0}^{n-1} R_j + r_n \left[1 - \frac{\alpha}{t} - \frac{2(n+r_n)}{t}\right]}{z-t} \\ &+ \frac{1}{z} \left[n - \sum_{j=0}^{n-1} R_j + (n+r_n) \left(\frac{2r_n}{t} - 1 \right) + \frac{\alpha r_n}{t} \right] \end{aligned} \quad (3.11)$$

and

$$\begin{aligned} \beta_n A_n(z) A_{n-1}(z) &= \frac{\beta_n R_n R_{n-1}}{(z-t)^2} + \frac{\beta_n (1-R_n)(1-R_{n-1})}{z^2} \\ &+ \frac{1}{t} \left(\frac{1}{z-t} - \frac{1}{z} \right) \beta_n [(1-R_n)R_{n-1} + (1-R_{n-1})R_n]. \end{aligned} \quad (3.12)$$

Now (S'_2) implies

Lemma 7 *For a fixed t , the quantities r_n , R_n , β_n satisfy the equations*

$$r_n^2 = \beta_n R_n R_{n-1} \quad (3.13)$$

$$(n+r_n)(n+\alpha+r_n) = \beta_n (1-R_n)(1-R_{n-1}) \quad (3.14)$$

$$\sum_{j=0}^{n-1} R_j + r_n \left[1 - \frac{\alpha}{t} - \frac{2(n+r_n)}{t} \right] = \frac{\beta_n}{t} [(1-R_n)R_{n-1} + (1-R_{n-1})R_n]. \quad (3.15)$$

Proof: The equations (3.13)—(3.15) are obtained by equating residues of (S'_2) . \square

In the next Lemma an expression is found for β_n in terms of r_n and R_n .

Lemma 8 *In terms of r_n and R_n , β_n the off-diagonal recurrence coefficient reads*

$$\beta_n = \frac{1}{1-R_n} \left[r_n(2n+\alpha) + n(n+\alpha) + \frac{r_n^2}{R_n} \right]. \quad (3.16)$$

Proof: We eliminate $\beta_n R_n R_{n-1}$ from (3.13) and (3.14) to find,

$$r_n(2n + \alpha) + n(n + \alpha) = \beta_n(1 - R_n - R_{n-1}) \quad (3.17)$$

$$= \beta_n(1 - R_n) - \frac{r_n^2}{R_n}. \quad (3.18)$$

In the last step we have used (3.13) to replace $\beta_n R_{n-1}$ by r_n^2/R_n . \square

We note that $B > 0$ can always be satisfied for the proper range of λ .

The equation (3.9) states that α_n is linear in R_n up to a linear form in n , together with (3.10) and (3.16), when combined with say, (3.13) provide us with a pair of non-linear difference equations satisfied by r_n and R_n . We state this in the next theorem.

Theorem 1 *The quantities r_n and R_n satisfy the difference equations;*

$$r_{n+1} + r_n = R_n(t - 2n - \alpha - 1 - tR_n) \quad (3.19)$$

$$r_n^2 \left(\frac{1}{R_n R_{n-1}} - \frac{1}{R_n} - \frac{1}{R_{n-1}} \right) = r_n(2n + \alpha) + n(n + \alpha) \quad (3.20)$$

with the "initial" conditions,

$$r_0(t) = 0 \quad (3.21)$$

$$R_0(t) = \frac{B t^\alpha e^{-t}}{h_0(t)} \quad (3.22)$$

$$h_0(t) = \left(1 - \frac{\beta}{2}\right)^\lambda \Gamma(1 + \alpha) + \left[\left(1 + \frac{\beta}{2}\right)^\lambda - \left(1 - \frac{\beta}{2}\right)^\lambda \right] \int_t^\infty x^\alpha e^{-x} dx. \quad (3.23)$$

Proof: This is simply a restatement of (3.10) and (3.13) with (3.9) and (3.16) \square

We shall see that (S'_2) automatically performs finite sums in "local" form, of the quantities R_n and α_n . This will be seen later to be relevant in the evaluation of the derivative of $\ln D_n(t)$ with respect to t and the derivation of the Painléve transcendent.

Theorem 2

$$t \sum_{j=0}^{n-1} R_j = -t r_n - n(n + \alpha) + \beta_n \quad (3.24)$$

$$\sum_{j=0}^{n-1} \alpha_j = -p_1(n) = \beta_n - t r_n. \quad (3.25)$$

Proof: From (3.15) we have,

$$\begin{aligned} t \sum_{j=0}^{n-1} R_j &= r_n [2(n + r_n) + \alpha - t] + \beta_n [R_n + R_{n-1} - 2R_n R_{n-1}] \\ &= r_n [2(n + r_n) + \alpha - t] + \beta_n [R_n + R_{n-1}] - 2r_n^2 \\ &= r_n [2(n + r_n) + \alpha - t] + \beta_n - r_n(2n + \alpha) - n(n + \alpha) - 2r_n^2 \\ &= -t r_n - n(n + \alpha) + \beta_n. \end{aligned} \quad (3.26)$$

The second equality of (3.26) follows from (3.13) and the third equality follows from (3.17). The equation (3.25) follows from (3.9) and the second equality of (2.10). \square

4 $P_V(0, -\frac{\alpha^2}{2}, 2n+1+\alpha, -\frac{1}{2})$

In this section we shall discover which of the auxiliary quantities defined as the residues of the rational functions $A_n(z)$ and $B_n(z)$ is a Painléve transcendent.

This will be obtained from a pair of Toda equations which shows that the Hankel determinant is the τ -function and these when suitably combined with the difference equations produce our P_V .

Taking the derivative of $h_n(t)$ with respect to t , we find

$$\frac{d}{dt} \ln h_n(t) = -B w_0(t) \frac{\{P_n(t, t)\}^2}{h_n(t)} = -R_n(t), \quad (4.1)$$

and consequently we have the Theorem

Theorem 3

$$\begin{aligned} -t \frac{d}{dt} \ln D_n(t) &= -t \sum_{j=0}^{n-1} \frac{d}{dt} \ln h_j(t) \\ &= t \sum_{j=0}^{n-1} R_j = -\mathbf{p}_1(n, t) - n(n + \alpha). \end{aligned} \quad (4.2)$$

Proof: The proof is obvious. \square

The next lemma gives the derivative of $\mathbf{p}_1(n, t)$ with respect to t .

Lemma 9

$$\frac{d}{dt} \mathbf{p}_1(n, t) = r_n(t). \quad (4.3)$$

Proof: Note the t dependence of $\mathbf{p}_1(n, t)$. Taking a derivative of

$$0 = \int_0^\infty P_n(x) P_{n-1}(x) w_J(x, t) w_0(x) dx,$$

with respect to t , produces,

$$\begin{aligned} 0 &= -B w_0(t) P_n(t, t) P_{n-1}(t, t) + \int_0^\infty \left[\frac{d}{dt} \mathbf{p}_1(n, t) x^{n-1} + \dots \right] P_{n-1}(x) w_J(x, t) w_0(x) dx \\ &= -B w_0(t) P_n(t, t) P_{n-1}(t, t) + h_{n-1} \frac{d}{dt} \mathbf{p}_1(n, t). \end{aligned}$$

and (4.3) follows. \square

We expect $D_n(t)$ to satisfied the Toda molecule equation [25] and this should indicate the emergence of a Painléve transcendant. The question that we will address is "Which quantity is satisfied by this particular Painléve transcendant?"

Theorem 4 *The Hankel determinant $D_n(t)$ satisfy the following differential-difference or the Toda molecule equation [25],*

$$t^2 \frac{d^2}{dt^2} \ln D_n(t) = -n(n + \alpha) + \frac{D_{n+1}(t)D_{n-1}(t)}{D_n^2(t)}. \quad (4.4)$$

Proof: Taking a derivative of (4.2) with respect to t and (4.3) imply

$$\frac{d}{dt} \left(t \frac{d}{dt} \ln D_n(t) \right) = r_n.$$

Now substitute r_n given above into (3.24) to find,

$$\begin{aligned} t \sum_{j=0}^{n-1} R_j &= -t \frac{d}{dt} \left[t \frac{d}{dt} \ln D_n(t) \right] - n(n + \alpha) + \beta_n. \\ &= -t \frac{d}{dt} \ln D_n(t), \end{aligned}$$

where the last equality comes from (4.2). The equation (4.4) follows if we recall

$$\beta_n = \frac{h_n}{h_{n-1}} = \frac{D_{n+1}D_{n-1}}{D_n^2},$$

since $D_n = h_0 \dots h_{n-1}$. \square

We now state a pair of somewhat non-standard Toda equations.

Lemma 10 *The recurrence coefficients α_n and β_n satisfy for $n \in \{1, 2, \dots\}$ the differential-difference equations*

$$\beta'_n(t) = (R_{n-1} - R_n)\beta_n \quad (T_1)$$

$$\alpha'_n(t) = r_n - r_{n-1}, \quad (T_2)$$

with $r_0(t)$ and $R_0(t)$ given by (3.22) and (3.23) respectively.

Proof: These equations are an immediate consequence of (4.1), (2.11), (4.3) and the first equality (2.10). \square

To discover the P_V of our problem. We first state two preliminary lemmas describing the t evolution of r_n and R_n .

Lemma 11 *For a fixed n , $R_n(t)$ satisfies the Riccati equation,*

$$tR'_n = 2r_n + (2n + \alpha - t + tR_n)R_n. \quad (4.5)$$

Proof: We begin with (T_2) and replace r_{n+1} by $R_n(t - \alpha_n) - r_n$. See (3.7). This leaves

$$\alpha'_n = 2r_n - (t - \alpha_n)R_n$$

After eliminating α_n in favor of R_n with (3.9) we have (4.5). \square

Lemma 12 For a fixed n , $r_n(t)$ satisfy the Riccati equation,

$$tr'_n = \frac{1 - 2R_n}{R_n(1 - R_n)} (r_n)^2 - (2n + \alpha) \frac{R_n r_n}{1 - R_n} - n(n + \alpha) \frac{R_n}{1 - R_n}. \quad (4.6)$$

Proof: By equating (3.24) to the last equality of (4.2), we find

$$\mathbf{p}_1(n, t) = tr_n - \beta_n.$$

Taking a derivative of the above equation with respect to t and noting (4.3) we see that

$$\begin{aligned} tr'_n &= \beta'_n \\ &= [R_{n-1} - R_n] \beta_n \\ &= \frac{r_n^2}{R_n} - \beta_n R_n, \end{aligned}$$

and use have been made of (T_2) and (3.13) to obtain the last two equalities. The equation (4.6) follows if we express β_n in terms of r_n and R_n using (3.16). \square

The next theorem shows that R_n is up to a linear fractional transformation a particular P_V .

Theorem 5 The quantity

$$S_n(t) := 1 - \frac{1}{R_n(t)}, \quad (4.7)$$

satisfies

$$S''_n = \frac{3S_n - 1}{2S_n(1 - S_n)} (S'_n)^2 - \frac{S'_n}{t} - \frac{\alpha^2}{2} \frac{(S_n - 1)^2}{t^2 S_n} + (2n + 1 + \alpha) \frac{S_n}{t} - \frac{1}{2} \frac{S_n(S_n + 1)}{S_n - 1}. \quad (4.8)$$

which is $P_V(0, -\alpha^2/2, 2n + 1 + \alpha, -1/2)$.

In terms of the recurrence coefficient $\alpha_n(t)$, we have,

$$S_n(t) = \frac{\alpha_n(t) - (2n + \alpha + 1) - t}{\alpha_n(t) - (2n + \alpha + 1)}. \quad (4.9)$$

Proof: Eliminate $r_n(t)$ from (4.5) and (4.6) and with $R_n = 1/(1 - S_n)$ gives (4.8). We have followed the convention of [14]. \square

Remark 3. Note that for $n = 0$, (4.8) is satisfied by

$$S_0(t) = 1 - \frac{1}{R_0(t)},$$

where $R_0(t)$ is given by (3.22) and (3.23) and ultimately in terms of an Incomplete Gamma function— a special case of the Kummer function of the second kind. Furthermore, since $r_0(t) = 0$, it can be verified that $R_0(t)$ also satisfy (4.5) at $n = 0$.

We may express the logarithmic derivative of $D_n(t)$ with respect to t in the so-called Jimbo-Miwa-Okamoto σ form. This is described in the next theorem.

Theorem 6 *Let*

$$H_n(t) := t \frac{d}{dt} \ln D_n(t), \quad (4.10)$$

then

$$(tH_n'')^2 = 4(H_n')^2[H_n - n(n + \alpha) - tH_n'] + [(2n + \alpha - t)H_n' + H_n]^2. \quad (4.11)$$

Proof: First we express $r_n(t)$ and $\beta_n(t)$ in terms of H_n and its derivatives. From (3.24) and (4.2) we have

$$\begin{aligned} -H_n &= -t r_n + \beta_n - n(n + \alpha) \\ &= -\mathbf{p}_1(n, t) - n(n + \alpha). \end{aligned} \quad (4.12)$$

Taking a derivative of (4.2) with respect to t and recalling (4.3) we have

$$r_n = H_n', \quad (4.13)$$

and with the first equality of (4.12) and (4.13), we find,

$$\beta_n = tH_n' - H_n + n(n + \alpha). \quad (4.14)$$

Now a derivative of (4.14) with respect to t and (T_1) gives

$$\begin{aligned} (tH_n')' - H_n' &= tH_n'' \\ &= \beta_n' = (R_{n-1} - R_n)\beta_n \\ &= \frac{r_n^2}{R_n} - \beta_n R_n. \end{aligned} \quad (4.15)$$

Here we have made use of (3.13) to arrive at the last equality. Therefore we have a quadratic equation in R_n ;

$$\frac{r_n^2}{R_n} - \beta_n R_n = tH_n''. \quad (4.16)$$

There is another quadratic equation in R_n which is a restatement of (3.16);

$$\frac{r_n^2}{R_n} + \beta_n R_n = \beta_n - (2n + \alpha)r_n - n(n + \alpha). \quad (4.17)$$

Now we solve for R_n and $1/R_n$ from (4.16) and (4.17) and find

$$\begin{aligned} \frac{2r_n^2}{R_n} &= \beta_n - (2n + \alpha)r_n - n(n + \alpha) + tH_n'' \\ 2\beta_n R_n &= \beta_n - (2n + \alpha)r_n - n(n + \alpha) - tH_n''. \end{aligned}$$

The equation (4.10) follows from the product of the above two equations,

$$4\beta_n r_n^2 = [\beta_n - (2n + \alpha)r_n - n(n + \alpha)]^2 - (tH_n'')^2,$$

and (4.13) and (4.14). \square

Incidentally R_n has two alternative representations,

$$R_n = \frac{tH_n'' + (2n + \alpha - t)H_n' + H_n}{2[H_n - n(n + \alpha) - tH_n']} \quad (4.18)$$

$$\frac{1}{R_n} = \frac{tH_n'' - (2n + \alpha - t)H_n' - H_n}{2(H_n')^2}. \quad (4.19)$$

The "discrete" structure inherited from the recurrence relations (2.7), induces a discrete analog of the σ form, namely, a non-linear second order difference equation in n satisfied by H_n for a fixed t ; we believe such a discrete form is new and may have been missed in previous similar studies perhaps because the recurrence relations were not sufficiently exploited. We note here that our derivation of (4.11) bypasses a third order equation and without having to identify a first integral which reduces the order by one.

We note also that equation (4.11) was first discovered by Tracy and Widom in [27] (which in our problem corresponds to $A = 0$ and $B = 1$) and just as was done in their paper for the Hermite case one can also rescale to obtain the Painlevé III equation corresponding to the Bessel kernel or "hard edge scaling". We change variables $t \rightarrow s/4n$, $H_n \rightarrow \sigma$ use (4.11) and keep only the highest order terms to obtain

$$(s\sigma'')^2 = 4\sigma(\sigma')^2 - 4s(\sigma')^3 - s(\sigma')^2 + \sigma\sigma' + \alpha^2(\sigma')^2.$$

Finally, we point out that the above analysis shows that the resolvent kernel used in the Tracy-Widom approach can be directly related to the orthogonal polynomials defined on (t, ∞) . In fact, if we denote $\tilde{R}(t, t)$ as the resolvent kernel defined in [27] then

$$t\tilde{R}(t, t) = H_n(t) = -tr_n - n(n + \alpha) + \beta_n.$$

Thus

$$t\tilde{R}(t, t) = -t \frac{Bw_0 P_n(t, t) P_{n-1}(t, t)}{h_{n-1}(t)} - n(n + \alpha) + \frac{h_n(t)}{h_{n-1}(t)}.$$

The term β_n can also be written using (3.16). In addition, we have that

$$\frac{d}{dt} \left(t\tilde{R}(t, t) \right) = r_n = \frac{Bw_0 P_n(t, t) P_{n-1}(t, t)}{h_{n-1}(t)}.$$

In other words we have found an identity for the resolvent kernel in terms of the values at the end points of the normalized orthogonal polynomials.

Theorem 7 *The auxiliary quantities R_n and r_n are expressed in terms H_n and $H_{n\pm 1}$ as follows:*

$$t R_n = H_n - H_{n+1} \quad (4.20)$$

$$t r_n = \frac{[H_n - n(n + \alpha)](t + H_{n+1} - H_{n-1}) + tn(n + \alpha)}{t + H_{n+1} - H_{n-1} - 2n - \alpha}. \quad (4.21)$$

The discrete analog of the σ form satisfied by H_n results from the substitution of (4.20) and (4.21) into

$$(t r_n)^2 = [n(n + \alpha) + t r_n - H_n][(t R_n)^2 + t R_n(H_{n+1} + H_{n-1} - 2H_n)]. \quad (4.22)$$

Proof: Taking a first order difference on the second equality of (4.2) together with (2.10) and (3.9) implies (4.20).

We re-write (3.24) gives

$$\beta_n = n(n + \alpha) + t r_n - H_n. \quad (4.23)$$

We will now find another equation expressing β_n in terms r_n , R_n , H_n , $H_{n\pm 1}$. Taking a first order difference on (4.20) gives,

$$t(R_n - R_{n-1}) = 2H_n - H_{n+1} - H_{n-1}.$$

Now multiply the above equation by R_n and make use of (3.13) we find

$$tR_n^2 - \frac{tr_n^2}{\beta_n} = (2H_n - H_{n+1} - H_{n-1})R_n,$$

and therefore

$$\frac{1}{\beta_n} = \frac{tR_n^2 - (2H_n - H_{n+1} - H_{n-1})R_n}{t r_n^2}. \quad (4.24)$$

Therefore the product of (4.23) and (4.24) implies (4.21), which leaves us the job of finding a further expression of r_n in terms of H_n and $H_{n\pm 1}$. For this purpose we rewrite (3.17) as

$$\beta_n(1 - R_n - R_{n-1}) = (2n + \alpha)r_n + n(n + \alpha).$$

Now substitute β_n given in (4.23) into the above resulting a linear equation in r_n ;

$$r_n[(t - tR_n - tR_{n-1}) - 2n - \alpha] = [H_n - n(n + \alpha)](1 - R_n - R_{n-1}) + n(n + \alpha).$$

With $t R_n$ as in (4.20) we have (4.21).

We summarize our results in the next theorem

Theorem 8 Let $D_n(t)$ be the Hankel determinant associated with the Laguerre weight perturbed by a jump factor, and

$$H_n(t) := t \frac{d}{dt} \ln D_n(t).$$

Then the recurrence coefficients are

$$\alpha_n(t) - (2n + \alpha + 1) = \frac{t^2 H_n'' + [(2n + \alpha)t - t^2] H_n' + t H_n}{2[H_n - n(n + \alpha) - t H_n']} \quad (4.25)$$

$$\beta_n(t) - n(n + \alpha) = t H_n' - H_n, \quad (4.26)$$

where $2n+1+\alpha$ and $n(n+\alpha)$ are the "unperturbed" recurrence coefficients and H_n satisfies a non-linear differential equation in Jimbo-Miwa-Okamoto σ form,

$$(t H_n'')^2 = 4(H_n')^2 [H_n - n(n + \alpha) - t H_n'] + [(2n + \alpha - t) H_n' + H_n]^2.$$

For the same H_n , the recurrence coefficients are

$$\alpha_n(t) - (2n + \alpha + 1) = H_n - H_{n+1} \quad (4.27)$$

$$\beta_n(t) - n(n + \alpha) = \frac{H_n(2n + \alpha) - n(n + \alpha)(H_{n+1} - H_{n-1})}{t + H_{n+1} - H_{n-1} - 2n - \alpha}, \quad (4.28)$$

where H_n satisfies the discrete σ form of a non-linear difference equation,

$$\begin{aligned} & \left\{ \frac{[H_n - n(n + \alpha)](t + H_{n+1} - H_{n-1}) + t n(n + \alpha)}{t + H_{n+1} - H_{n-1} - 2n - \alpha} \right\}^2 \\ &= \left\{ \frac{(2n + \alpha)[H_n - n(n + \alpha)] + t n(n + \alpha)}{t + H_{n+1} - H_{n-1} - 2n - \alpha} \right\} (H_n - H_{n+1})(H_{n-1} - H_n). \end{aligned} \quad (4.29)$$

Note that since $\alpha_n(t)$ and $\beta_n(t)$ have two alternative representations, $H_n(t)$ satisfies two more differential-difference equations, $(4.25) = (4.27)$ and $(4.26) = (4.28)$.

We end this paper with a discussion on the relationship between our P_V and the difference equations (3.19) and (3.20). We would like to thank the second referee for supplying us the background material part of which is reproduced here.

The fifth Painlevé equation $P_V(a, b, c, d = -1/2)$:

$$y'' = \left(\frac{1}{2y} + \frac{1}{y-1} \right) (y')^2 - \frac{y'}{t} + \frac{(y-1)^2}{t^2} \left(ay + \frac{b}{y} \right) + c \frac{y}{t} + d \frac{y(y+1)}{y-1}, \quad ' = \frac{d}{dt}$$

is equivalent to the Hamiltonian system \mathcal{H}_V :

$$q' = \frac{\partial \mathbf{H}}{\partial p}, \quad tp' = -\frac{\partial \mathbf{H}}{\partial q},$$

with the time-dependent Hamiltonian $\mathbf{H} = \mathbf{H}(p, q, t)$:

$$t\mathbf{H} = p(p+t)q(q-1) + \alpha_2 qt - \alpha_3 pq - \alpha_1 p(q-1),$$

where

$$a = \frac{\alpha_1^2}{2}, \quad b = -\frac{\alpha_3^2}{2}, \quad c = \alpha_0 - \alpha_2, \quad d = -\frac{1}{2}, \quad \alpha_0 := 1 - \alpha_1 - \alpha_2 - \alpha_3$$

and

$$y = 1 - \frac{1}{q}.$$

The Hamiltonian structure was studied in [22] and the τ -function is defined such that

$$\frac{d}{dt} \ln \tau = \mathbf{H}.$$

The extended affine Weyl group $W(A_3^{(1)}) = \langle s_0, s_1, s_2, s_3, \pi \rangle$ of the Weyl group type $A_3^{(1)}$ acts as bi-rational symmetries on P_V and induces Backlund transformations on the solutions of P_V . Here the s'_i 's and π are the generators. See [22] for the study of Weyl group actions on P_V .

For example, the action of s_0 :

$$\begin{aligned} s_0\{\alpha_0, \alpha_1, \alpha_2, \alpha_3\} &= \{-\alpha_0, \alpha_1 + \alpha_0, \alpha_2, \alpha_3 + \alpha_0\}, \\ s_0(q) &= q + \frac{\alpha_0}{p+t}, \\ s_0(p) &= p, \end{aligned}$$

leaves P_V or the Hamiltonian system \mathcal{H}_V invariant. We refer the readers to [22, 28] for information on Weyl group actions and [(4.3), [28]] which lists the bi-rational transformations.

To proceed further, consider a parallel transformation $l = (s_2 s_3 \pi)^2 \in W(A_3^{(1)})$:

$$l : \vec{\alpha} = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \longmapsto \vec{\alpha} + (1, 0, -1, 0).$$

From a direct computation, we may verify that, the variables q and $r := pq(q-1)$ satisfies the following system of difference equations:

$$l(r) + r = q(\alpha_2 - \alpha_0 + t - tq) - \alpha_1 \tag{4.30}$$

$$\left(\frac{1}{q} - 1\right) \left(\frac{1}{l^{-1}(q)} - 1\right) = \frac{(r - \alpha_2)(r - \alpha_2 - \alpha_3)}{r(r + \alpha_1)}, \tag{4.31}$$

these seems to the $d - P_{\text{III}}$ of [205, [23]] in disguise.

In terms of \mathbf{H} our auxiliary parameter r reads

$$r = \frac{d}{dt} (t\mathbf{H}). \tag{4.32}$$

In our problem we have $P_V(0, -\alpha^2/2, 2n+1+\alpha, -1/2)$, which implies

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (1+n, 0, -n-\alpha, \alpha).$$

If

$$\begin{aligned} l^n(q) &= q_n =: R_n \\ l^n(r) &=: r_n, \end{aligned}$$

for $n \in \{0, 1, 2, \dots\}$, then a direct computation shows that (4.33) and (4.34) are

$$r_{n+1} + r_n = R_n(-\alpha - 2n - 1 + t - tR_n), \quad (4.33)$$

$$\left(\frac{1}{R_n} - 1\right) \left(\frac{1}{R_{n-1}} - 1\right) = \frac{(r_n + n + \alpha)(r_n + n)}{r_n^2}, \quad (4.34)$$

are equivalent to (3.19) and (3.20) respectively.

We should like to mention here that (3.19) and (3.20) and other equations are derived entirely from orthogonality and the immediate consequence—the recurrence relations.

In view of (3.32) we see that the logarithmic derivative of the generating function $G(n, t) = D_n(t)$ is the τ -function of our P_V . We end this paper with the final remark: the equation (4.4) is essentially the same as the Toda equation among a τ -sequence discovered by Okamoto [22].

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